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Stewartson layers in transient rotating fluid flows

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The formation and transient dynamics of Stewartson layers are examined herein. The results are applied to a transient rotating fluid flow for which (i) Stewartson layers are presumed to be important, and (ii) the Ekman layers are found to be non-divergent, and hence unable to induce the crucial secondary circulation occurring in the spin-up process. The steady state is nevertheless reached after a spin-up time.

1. Introduction

The transient motions occurring in a fluid as it adjusts from one state of rigid body rotation to another were examined by Greenspan & Howard (1963) (hereafter this paper is referred to as G & H). The importance of this spin-up process stems from the fact that it is typical of various transient processes in rotating fluid dynamics, particularly in so far as the duration of the transient and the role of the Ekman layers are concerned. More specifically, G & H found that the spin-up time is of the order of $(L^2/\nu\Omega)^{\frac{1}{2}}$, where Ω is the angular velocity, ν the kinematic viscosity of the fluid and L a characteristic height of the container. The Ekman layers are able, by means of their suction, to induce secondary circulations which affect profoundly the vorticity and angular momentum fields. By this mechanism, the Ekman layers strongly control the transient process.

In the spin-up process, Stewartson layers are either passive or altogether absent. As a result, their transient behaviour was very briefly mentioned by G & H, and their formation was not discussed. However, there could be transient rotating fluid flows in which these layers might play an important role. The following problem which we propose to investigate falls into this category.

A circular, cylindrical annulus filled with fluid rotates as a rigid body around a vertical axis. At a given time, a flow is induced by injecting and withdrawing fluid through the inner and outer vertical boundaries respectively.

Apart from the fact that the above problem calls for transient Stewartson layers and that it is of some geophysical relevance (Barcilon 1967*a*), it is also of interest on account of the Ekman layers which arise. Indeed, as we shall presently see, these layers are non-divergent throughout the transient process and therefore the crucial suction mechanism involved in the spin-up is absent. As a result, it is not altogether clear whether the spin-up time characterizes the duration of the transient.

Before answering these questions, we shall have to consider the transient

Stewartson layers *per se*. In particular, we shall have to examine their formation time and their transient dynamics.

This, together with an approximate solution of the above-mentioned problem, will constitute the purpose of the present paper. As far as possible, we shall carry out the investigation of the transient Stewartson layers independently of the transient source–sink flow, so that this section of the paper could be read separately.

2. Basic equations

In the present paper we shall restrict our attention to flows which are small deviations from a state of rigid rotation about the vertical. In other words, the Rossby number

$$\epsilon = U/\Omega L,$$

where U is a characteristic velocity, is assumed to be small. The length L is the height of the container: consequently, if z is the dimensionless vertical co-ordinate, the horizontal boundaries are at $z = 0$ and $z = 1$.

The fluid is assumed to be homogeneous, incompressible and viscous. The linearized dimensionless equations of motion written in the frame in which the fluid is originally at rest are

$$\left.\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} + 2\hat{\mathbf{k}} \times \mathbf{q} &= -\nabla p + E\nabla^2\mathbf{q}, \\ \nabla . \mathbf{q} &= 0, \end{aligned}\right\} \tag{1}$$

where p is the dimensionless pressure, $\mathbf{q}$ the dimensionless velocity and $\hat{\mathbf{k}}$ is a unit vector in the vertical direction. The parameter E, the Ekman number, is defined thus:

$$E = \nu/\Omega L^2,$$

and represents a measure of the viscous force relative to the Coriolis force. Throughout this paper E is assumed to be very small. The time is scaled with respect to the rotation frequency Ω; consequently the dimensionless spin-up time is $E^{-\frac{1}{2}}$.

3. Transient Stewartson layers

Steady Stewartson layers are vertical boundary layers which arise as an indirect consequence of the strong constraint placed on the dynamics of rotating fluids by the Taylor–Proudman theorem (Taylor 1923). They occur either along vertical boundaries, or as detached (free) shear layers straddling the vertical vortex sheets deduced from the inviscid dynamics (Stewartson 1957). In general, they have a double structure which consists of an inner layer of thickness $E^{\frac{1}{3}}$ and of an outer layer of thickness $E^{\frac{1}{4}}$. Their detailed structure is however strongly dependent upon the problem under consideration and the role they must fulfil: adjustment of tangential velocity (Stewartson 1957; Jacobs 1964), smearing-out of z-dependence (Barcilon 1967*b*), etc. As a result, our understanding of these layers has evolved, piecemeal, from a collection of different problems in which they occurred. The superficial diversity of the Stewartson layers stems from the fact that the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers behave as separate entities which can arise in various combinations.

Bearing in mind the above remarks, we shall try to obtain a general picture of the transient Stewartson layers by considering two specific and rather distinct problems in which they arise, and by treating as much as possible the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layer separately. Furthermore, we shall emphasize the basic features of the transient process rather than the structural details of these layers.

In this section, we examine the case in which Stewartson layers are generated by the shearing motion of the vertical wall of a circular cylinder. This problem is chosen because of its relative simplicity and because of its suitability to vorticity arguments. More specifically, we consider the transient Stewartson layers generated along the side wall of a circular cylinder by applying an impulsive zonal velocity to this side wall. Since the thicknesses of the boundary layers are assumed to be much smaller than the radius of the cylinder, we can neglect the effects of curvature and formulate the problem as follows.

Consider the fluid region $x \geqslant 0$, $-\infty < y < \infty$, $0 < z < 1$ bounded by three rigid walls. At time $t = 0$ the vertical boundary $x = 0$ is moved in its own plane with a velocity $V_0(z)\,\mathbf{j}$ independent of y, where $\mathbf{j}$ is a unit vector in the y-direction. It is convenient (Barcilon 1967*b*) to decompose $V_0(z)$ into two parts, viz.

$$V_0(z) = \langle V \rangle + \mathscr{V}(z), \tag{2}$$

where

$$\langle V \rangle = \int_0^1 V_0(z)\,dz \tag{3}$$

is the average 'zonal' velocity of the wall. We shall consider the Stewartson layers induced by $\mathscr{V}(z)$ and $\langle V \rangle$ separately and in that order.

3.1. $E^{\frac{1}{3}}$ *layer*

The vortex sheet of variable strength caused by the shearing motion of the boundary diffuses into the fluid and gives rise to a Rayleigh layer. In the initial phase of the transient process, the y-component of the velocity in this Rayleigh layer is a function of the similarity variable x^2/Et; its z-dependence is carried parametrically and is essentially that of the boundary velocity, except in the immediate neighbourhood of the horizontal walls along which Ekman layers are produced. After a time of $O(1)$, i.e. after a few revolutions, the thickness of the Rayleigh layer is of $O(E^{\frac{1}{2}})$ and the Ekman layers are completely formed (see G & H). This can be regarded as the first part of the transient process.

The Stewartson layer starts to form after this first phase is over. During the second half of the transient, we can (i) consider the Ekman layers as quasi-steady and (ii) use the Ekman compatibility conditions, viz.

$$w = \mp \frac{E^{\frac{1}{2}}}{2}\frac{\partial v}{\partial x} \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{4}$$

since, from then on, the Rayleigh layer is thicker than $E^{\frac{1}{2}}$. Sacrificing the exact description of the first half of the transient, we shall assume that the Ekman layers are quasi-steady and that the compatibility conditions can be used throughout the *entire* transient. This procedure, which was successfully used by G & H, has the advantage of reducing considerably the mathematical difficulties inherent in time-dependent rotating fluid flow problems.

Using the scaling appropriate for the steady state, viz.

$$\left.\begin{aligned} u &= E^{\frac{1}{3}}\tilde{u}(\xi, z, T), \\ v &= \tilde{v}(\xi, z, T), \\ w &= \tilde{w}(\xi, z, T), \\ p &= E^{\frac{1}{3}}\tilde{p}(\xi, z, T). \end{aligned}\right\} \tag{5}$$

where

$$\xi = E^{-\frac{1}{3}}x, \tag{6}$$

it is easy to see after substitution in the equations of motion that the long time variable must be

$$T = E^{\frac{1}{3}}t. \tag{7}$$

Consequently, the transient $E^{\frac{1}{3}}$ layer equations are

$$\left.\begin{aligned} -2\tilde{v} &= -\tilde{p}_{\xi}, \\ \tilde{v}_T + 2\tilde{u} &= \tilde{v}_{\xi\xi}, \\ \tilde{w}_T &= -\tilde{p}_z + \tilde{w}_{\xi\xi}, \\ \tilde{u}_{\xi} + \tilde{w}_z &= 0. \end{aligned}\right\} \tag{8}$$

The balance in the direction normal to the boundary layer is quasi-geostrophic. The boundary conditions associated with the above equations are

$$\left.\begin{aligned} \tilde{w} = 0 \quad &\text{at} \quad z = 0, 1, \\ \tilde{u} = \tilde{v} - \mathscr{V}(z) = \tilde{w} = 0 \quad &\text{at} \quad \xi = 0, \\ \tilde{u}, \tilde{v}, \tilde{w} \to 0 \quad &\text{as} \quad \xi \to \infty. \end{aligned}\right\} \tag{9a}$$

The initial conditions for (8) should reflect the state of the fluid at $t \sim 1$ rather than at $t = 0$. However, after the first phase of the transient is over, the motion is confined to distances smaller than $E^{\frac{1}{2}}$; in other words, for all ξ's of $O(1)$ the fluid is at rest, and hence

$$\tilde{v} = \tilde{w} = 0 \quad \text{for} \quad T = 0. \tag{9b}$$

Integral representations of the solution of the above initial boundary-value problem can be obtained by means of Laplace transform and appropriate Fourier series in z. However, because these representations are lengthy and not very revealing, they will not be given here. Rather, let us discuss in physical terms the transient dynamics from the point of view of vorticity.

Eliminating $\tilde{u}$ and $\tilde{p}$ from (8), we essentially obtain the components of the vorticity equation in the y- and z-directions, viz.

$$\left.\begin{aligned} \frac{\partial}{\partial T}\left(\frac{\partial \tilde{w}}{\partial \xi}\right) + 2\frac{\partial \tilde{v}}{\partial z} &= \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \tilde{w}}{\partial \xi}\right), \\ \frac{\partial}{\partial T}\left(\frac{\partial \tilde{v}}{\partial \xi}\right) - 2\frac{\partial \tilde{w}}{\partial z} &= \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \tilde{v}}{\partial \xi}\right). \end{aligned}\right\} \tag{10}$$

On account of its vertical structure, the $\tilde{v}$-field created by the shearing motion of the boundary will tilt the vortex lines in the y-direction, thus creating vorticity in that direction. This y-component of vorticity will give rise to a circulation in the (x, z)-plane, i.e. will generate a $\tilde{u}$- and a $\tilde{w}$-field. In turn, the vertical structure of the $\tilde{w}$-field thus created will be responsible for a stretching of the vortex lines and hence a production of vertical vorticity. The cycle is then closed by noting that the vertical vorticity is related to the $\tilde{v}$-field (see figure 1). After

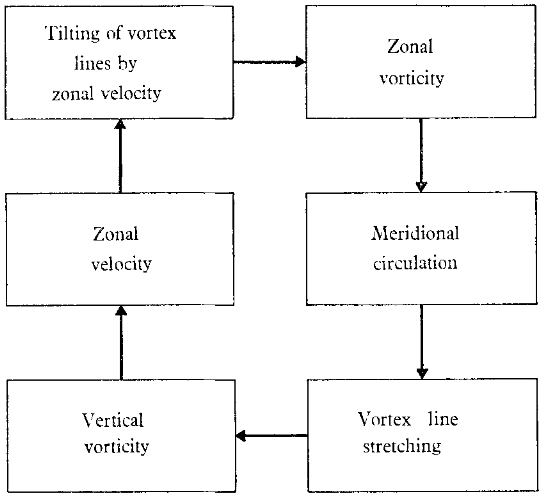

FIGURE 1. Schematic diagram of the transient $E^{\frac{1}{3}}$ layer dynamics. 'Zonal component' and 'Meridional circulation' refer to the horizontal component and to the circulation in vertical planes respectively parallel and perpendicular to the boundary-layer surface.

a time of $O(E^{-\frac{1}{3}})$, i.e. after a time shorter than the spin-up time, the exchange between the tangential and vertical components of vorticity is damped by viscosity and the steady state is reached. The $E^{\frac{1}{3}}$ layer, which is often required to 'shield' the interior region from boundary conditions whose z-dependence is incompatible with the Taylor–Proudman theorem, relies precisely on these variations for its existence.

A few remarks are worth making at this point. The flow induced by moving the boundary with a velocity $\mathscr{V}(z)$ whose vertical average is zero, is entirely confined to an $E^{\frac{1}{3}}$ layer. This lack of interaction between the $E^{\frac{1}{3}}$ layer and the interior is rather typical. Consequently, if the transient interior dynamics is to be influenced by Stewartson layers, this can only occur through the $E^{\frac{1}{4}}$ layer. Finally, the vorticity equations (10) are not altered when the fields have a y-dependence, i.e. when the boundary velocity $V_0(y, z)$ is a function of y and z; as a result, the above picture of the transient is still valid.

3.2. $E^{\frac{1}{4}}$ *layer*

Let us now consider the layer generated by the average velocity $\langle V \rangle$, which we expect to be made up mainly of an $E^{\frac{1}{4}}$ layer on account of the lack of z-dependence of the boundary condition.

The first part of the transient process is identical with that of the $E^{\frac{1}{3}}$ layer: after a few revolutions, the Rayleigh layer diffuses over a distance of $O(E^{\frac{1}{2}})$ and

Ekman layers are formed along the horizontal boundaries. Instead of being produced as a result of both the vortex-line tilting and the vertical pressure gradient, the vertical velocity is now induced by the Ekman-layer suction. This creates the usual vortex-line stretching and hence vertical vorticity which is diffused by viscous action. Sacrificing once more the exact description of the first phase of the transient, we shall again assume that the Ekman layers are quasi-steady and that the compatibility conditions are valid throughout the entire transient. Using the scaling of the various fields appropriate for the steady state, viz.

$$\left.\begin{aligned} u &= E^{\frac{1}{2}}\bar{u}(\eta, z, \tau), \\ v &= \bar{v}(\eta, z, \tau), \\ w &= E^{\frac{1}{4}}\bar{w}(\eta, z, \tau), \\ p &= E^{\frac{1}{4}}\bar{p}(\eta, z, \tau), \end{aligned}\right\} \tag{11}$$

where
$$\eta = E^{-\frac{1}{4}}x \tag{12}$$

is the stretched variable, we see after substitution in the equations of motion that the long time variable must be

$$\tau = E^{\frac{1}{2}}t. \tag{13}$$

Hence, the time necessary for the $E^{\frac{1}{4}}$ layer to form is the spin-up time. The equations for the transient $E^{\frac{1}{4}}$ layer are:

$$\left.\begin{aligned} -2\bar{v} &= -\bar{p}_\eta, \\ \bar{v}_\tau + 2\bar{u} &= \bar{v}_{\eta\eta}, \\ 0 &= -\bar{p}_z, \\ \bar{u}_\eta + \bar{w}_z &= 0. \end{aligned}\right\} \tag{14}$$

By means of arguments similar to those used for the $E^{\frac{1}{3}}$ layer, we can deduce that the initial condition appropriate for (14) is

$$\bar{v} = 0 \quad \text{at} \quad \tau = 0. \tag{15a}$$

The boundary conditions along the horizontal walls are simply the Ekman compatibility conditions

$$\bar{w} = \mp \tfrac{1}{2}\bar{v}_\eta \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{15b}$$

The usual exponential decay of the boundary-layer fields provides one of the two conditions in the η-direction, viz.

$$\bar{v} \to 0 \quad \text{as} \quad \eta \to \infty; \tag{15c}$$

the other condition is
$$\bar{v} = \langle V \rangle \quad \text{at} \quad \eta = 0, \tag{15d}$$

since, as shown in the appendix, there is no $O(1)$ 'zonal' velocity in the interior.

From (14) we can see that $\bar{p}$, and hence $\bar{u}$ and $\bar{v}$, are independent of z throughout the transient. The Ekman compatibility conditions (15b) and the continuity equation can be identically satisfied by writing $\bar{u}$, $\bar{v}$ and $\bar{w}$ in terms of a single function $\bar{\phi}$ of τ and η thus:

$$\left.\begin{aligned} \bar{u} &= \bar{\phi}, \\ \bar{v} &= \bar{\phi}, \\ \bar{w} &= -(z-\tfrac{1}{2})\bar{\phi}_\eta. \end{aligned}\right\} \tag{16}$$

The equation for $\overline{\phi}$, obtained by substituting the above expressions in the y-component of the momentum equation, is

$$\overline{\phi}_\tau + 2\overline{\phi} = \overline{\phi}_{\eta\eta}. \tag{17}$$

Expressions for the $E^{\frac{1}{4}}$ layer velocity fields can now be easily found, viz.

$$\left.\begin{aligned} \overline{u} = \overline{v} = \langle V\rangle \Big\{ &\exp[-\sqrt{2}\,\eta] - \tfrac{1}{2}\exp[-\sqrt{2}\,\eta]\operatorname{erfc}\left(\sqrt{(2\tau)} - \frac{\eta}{2\sqrt{\tau}}\right) \\ &+ \tfrac{1}{2}\exp[\sqrt{2}\,\eta]\operatorname{erfc}\left(\sqrt{(2\tau)} + \frac{\eta}{2\sqrt{\tau}}\right)\Big\}, \\ \overline{w} = \sqrt{2}(z-\tfrac{1}{2})\langle V\rangle \Big\{ &\exp[-\sqrt{2}\,\eta] - \tfrac{1}{2}\exp[-\sqrt{2}\,\eta]\operatorname{erfc}\left(\sqrt{(2\tau)} - \frac{\eta}{2\sqrt{\tau}}\right) \\ &- \tfrac{1}{2}\exp[\sqrt{2}\,\eta]\operatorname{erfc}\left(\sqrt{(2\tau)} + \frac{\eta}{2\sqrt{\tau}}\right)\Big\}. \end{aligned}\right\} \tag{18}$$

The expressions (18) do not represent the complete velocity fields, if only because $\overline{u}$ and $\overline{w}$ do not satisfy the no-slip condition. This feature is typical of the $E^{\frac{1}{4}}$ layer, which cannot occur all by itself and which must be coupled to an $E^{\frac{1}{3}}$ layer or to the interior, or both. In particular, whenever the side-wall driving produces an interior flow, the $E^{\frac{1}{4}}$ layer will interact with the interior. It is therefore already apparent that the spin-up time is the characteristic time scale even for interior flows controlled by Stewartson layers.

For the specific problem considered here, only an $E^{\frac{1}{3}}$ layer is required to adjust the $E^{\frac{1}{4}}$ layer fields as given by (18) (see the appendix). Its primary role is to 'close' the vertical mass transport. The flow in the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layer combination consists of pairs of re-circulating cells which are symmetric with respect to the plane $z = \frac{1}{2}$. After the steady state is reached, these layers are essentially identical with those originally discussed by Stewartson (1957) in connexion with the flow induced by two disks rotating in unison (the so-called 'symmetrical problem').† In the interior, the fluid is not spun up; in fact, it remains at rest throughout the entire transient process.

In conclusion, it is interesting to note that the formation times of the Ekman, $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ Stewartson layers are simply equal to the time required for the diffusion to be felt over distances equal to the thicknesses of these respective layers.

The results of the present paragraph suggest that valid approximate solutions of a transient rotating fluid flow problem can be obtained (i) by treating both the Ekman and $E^{\frac{1}{3}}$ layers as quasi-steady and (ii) by solving only the equations for the interior and $E^{\frac{1}{4}}$ layer fields, which are assumed to be functions of τ but not t or T. By means of this procedure, inertial waves are of course filtered out.

4. Transient flow due to a source–sink distribution

Let us consider a circular cylindrical annulus of unit height and of inner and outer radius a and b. This annulus is filled with fluid and is rotating rigidly about its vertical axis. At time $t = 0$, fluid is injected and withdrawn uniformly along

† The only difference lies in the fact that the layers considered by Stewartson were detached.

the inner and outer vertical walls, i.e. the boundary conditions along these walls are

$$\left.\begin{aligned} \mathbf{q} &= \hat{\mathbf{r}}E^{\frac{1}{2}} \quad \text{at} \quad r = a, \\ \mathbf{q} &= \frac{a}{b}\hat{\mathbf{r}}E^{\frac{1}{2}} \quad \text{at} \quad r = b, \end{aligned}\right\} \tag{19}$$

where $\hat{\mathbf{r}}$ is a unit vector in the radial direction; the factor $E^{\frac{1}{2}}$ is introduced purely for convenience.

We first consider the interior fields, which will be denoted by capitals. Guided by the steady-state solutions (Lewellen 1965), we look for transient solutions of the form

$$\left.\begin{aligned} U &= E^{\frac{1}{2}}U^{(0)}(r,z,\tau) + \dots, \\ V &= V^{(0)}(r,z,\tau) + \dots, \\ W &= E^{\frac{1}{2}}W^{(0)}(r,z,\tau) + \dots, \\ P &= P^{(0)}(r,z,\tau) + \dots. \end{aligned}\right\} \tag{20}$$

The zeroth-order interior dynamics is governed by the following equations:

$$\left.\begin{aligned} -2V^{(0)} &= -P_r^{(0)}, \\ V_\tau^{(0)} + 2U^{(0)} &= 0, \\ 0 &= -P_z^{(0)}, \\ (1/r)(rU^{(0)})_r + W_z^{(0)} &= 0. \end{aligned}\right\} \tag{21}$$

We therefore see that $P^{(0)}$, and hence $U^{(0)}$ and $V^{(0)}$, are independent of z and that $W^{(0)}$ is at most a linear function of z. The horizontal boundary conditions for (21) are the quasi-steady Ekman compatibility conditions

$$W^{(0)} = \mp \tfrac{1}{2}(1/r)(rV^{(0)})_r \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{22}$$

Introducing a stream-like function $\psi(r,\tau)$ such that

$$\left.\begin{aligned} U^{(0)} &= \psi, \\ W^{(0)} &= -(z-\tfrac{1}{2})(1/r)(r\psi)_r, \end{aligned}\right\} \tag{23}$$

we can deduce from (22) that

$$\psi = V^{(0)} + \frac{F(\tau)}{r}, \tag{24}$$

where $F(\tau)$ is an arbitrary function of τ. Substituting the above expressions for the velocity fields in the zonal momentum equation, we get an equation for $V^{(0)}$, viz.

$$V_\tau^{(0)} + 2V^{(0)} = -(1/r)F(\tau). \tag{25}$$

Recalling that the zonal velocity is initially equal to zero, we deduce that

$$V^{(0)} = -\frac{2}{r}e^{-2\tau}\int_0^\tau F(\lambda)\,e^{2\lambda}\,d\lambda, \tag{26}$$

and hence

$$U^{(0)} = -\frac{2}{r}e^{-2r}\int_0^\tau F(\lambda)\,e^{2\lambda}\,d\lambda + \frac{F(\tau)}{r}. \tag{27}$$

In order to determine the function $F(\tau)$ we must consider the transient $E^{\frac{1}{4}}$ layers on the inner and outer walls. The equations for these $E^{\frac{1}{4}}$ layers are identical with (14) provided η is interpreted as $E^{-\frac{1}{4}}(r-a)$ and $E^{-\frac{1}{4}}(b-r)$ for the inner and outer walls respectively. Consequently, we can avail ourselves of the expressions (16) and write the $E^{\frac{1}{4}}$ layer correction fields as

$$\left.\begin{aligned} u &= E^{\frac{1}{4}}\overline{\phi}(\eta,\tau), \\ v &= \quad \overline{\phi}(\eta,\tau), \\ w &= -E^{\frac{1}{4}}(z-\tfrac{1}{2})\frac{\partial\overline{\phi}}{\partial\eta}. \end{aligned}\right\} \tag{28}$$

For the inner wall, we are therefore led to solve the following equation

$$\overline{\phi}_\tau + 2\overline{\phi} = \overline{\phi}_{\eta\eta} \tag{29}$$

subject to the conditions

$$\left.\begin{aligned} &\overline{\phi} = 0 \quad \text{at} \quad \tau = 0, \\ &\overline{\phi} + U^{(0)} - 1 = \overline{\phi} + V^{(0)} = 0 \quad \text{at} \quad \eta = 0, \\ &\overline{\phi} \to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{30}$$

The boundary conditions at $\eta = 0$ must be compatible for a solution to exist. This compatibility condition, which is

$$U^{(0)} - 1 = V^{(0)} \quad \text{at} \quad r = a, \tag{31}$$

determines $F(\tau)$, viz.

$$F(\tau) = a. \tag{32}$$

The same result is obtained by examining the $E^{\frac{1}{4}}$ layer on the outer wall $r = b$. Using now a Laplace transform, we can easily solve for $\overline{\phi}$,

$$\overline{\phi} = -e^{-2\tau}\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\tau}}\right) + \tfrac{1}{2}\exp\left[\sqrt{2}\,\eta\right]\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\tau}} + \sqrt{(2\tau)}\right) + \tfrac{1}{2}\exp\left[-\sqrt{2}\,\eta\right]\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\tau}} - \sqrt{(2\tau)}\right), \tag{33}$$

and for the interior fields

$$\left.\begin{aligned} U^{(0)} &= (a/r)\,e^{-2\tau}, \\ V^{(0)} &= -(a/r)[1-e^{-2\tau}], \\ W^{(0)} &= 0. \end{aligned}\right\} \tag{34}$$

In addition to the above interior and $E^{\frac{1}{4}}$ layer fields, one would also have to introduce weak $E^{\frac{1}{3}}$ layers to adjust the vertical velocity along the side walls. As previously mentioned these layers can be treated as quasi-steady; the lengthy expressions for the velocity fields in these layers are not given here.

A fairly detailed picture of the transient process can now be inferred from (34). At the beginning of the transient, the flow is mainly radial, and comparable to what it would be if the container was not rotating. However, the protracted action of the Coriolis force deflects this radial flow to the right, thus inducing a clockwise zonal motion. All the while, these interior motions are adjusted to zero along the horizontal walls by means of Ekman layers. As the strength of the zonal flow increases, so does that of the $E^{\frac{1}{4}}$ layers which are needed to satisfy the no-slip conditions along the vertical boundaries. As a consequence, the $E^{\frac{1}{4}}$ layer along the inner wall feeds an increasingly large fraction of the injected fluid to the top and bottom Ekman layers; when the steady state is reached, the radial mass transport is entirely effected by the Ekman layers and the interior flow is purely zonal. Throughout the entire transient, and even in the steady state, the top and bottom Ekman layers are *non-divergent* in the interior.

5. Conclusion

The transient source–sink flow investigated here provides an example of a rotating fluid flow in which (i) transient Stewartson layers had to be taken into account and (ii) Ekman layers were found to be non-divergent. The fact that the steady state is reached after a spin-up time is therefore rather surprising. Indeed, in the spin-up process, the divergence of the Ekman layers is playing a crucial role and the spin-up time is intimately related to this suction mechanism.

The fact that the $E^{\frac{1}{4}}$ layer is formed over a spin-up time could, of course, be invoked to explain this result. However, this explanation is only partially correct. Recalling that the very existence of the $E^{\frac{1}{4}}$ layers depends on the Ekman layer suction, we are forced to the following conclusion. Either by direct action over the interior (as in the spin-up process), or indirectly through the $E^{\frac{1}{4}}$ layers (as in the present case), the Ekman layers are always important during the transient process.

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Appendix

We outline here how the solution to the problem considered in §3.2 can be obtained. In particular, we shall show that the fluid's interior is not spun up and discuss the nature of the $E^{\frac{1}{3}}$ layer required to adjust the velocity fields in the $E^{\frac{1}{4}}$ layer as given by (18).

Denoting once again the interior fields by capitals, and reverting to cylindrical co-ordinates, we look for solutions of the form

$$\left.\begin{aligned} U &= E^{\frac{1}{2}}U^{(0)}(r,z,\tau)+\ldots,\\ V &= V^{(0)}(r,z,\tau)+\ldots,\\ W &= E^{\frac{1}{2}}W^{(0)}(r,z,\tau)+\ldots,\\ P &= P^{(0)}(r,z,\tau)+\ldots. \end{aligned}\right\} \tag{A 1}$$

The equations for $U^{(0)}$, $V^{(0)}$ and $W^{(0)}$ are identical with (21) and consequently the representation of these fields in terms of a function ψ of r and τ is identical with that given in (23) and (24), viz.

$$\left.\begin{aligned} U^{(0)} &= \psi(r,\tau),\\ V^{(0)} &= -\psi(r,\tau)-(1/r)\,F(\tau),\\ W^{(0)} &= -(z-\tfrac{1}{2})\,(1/r)\,(r\psi)_r. \end{aligned}\right\} \tag{A 2}$$

However, since both $U^{(0)}$ and $V^{(0)}$ must be regular at $r=0$, we must set the function $F(\tau)$ equal to zero. As a result, the initial value problem for ψ is homogeneous and hence ψ, $U^{(0)}$, $V^{(0)}$ and $W^{(0)}$ are identically zero. The flow induced by spinning-up the side wall of a cylinder is therefore confined to the Stewartson layer.

Turning now to the $E^{\frac{1}{3}}$ layer, we obtain the magnitude of the first non-zero terms in the asymptotic series representation of the various fields, by requiring that the vertical mass flux in this layer be comparable to that in the $E^{\frac{1}{4}}$ layer. As we shall presently see, this requirement is equivalent to the no-slip condition for the radial velocity. With the same notations as before, the complete velocity fields are

$$\left.\begin{aligned} u &= E^{\frac{1}{2}}[\bar{u}+\ldots]+E^{\frac{1}{2}}[\tilde{u}+\ldots], \\ v &= \quad[\bar{v}+\ldots]+E^{\frac{1}{6}}[\tilde{v}+\ldots], \\ w &= E^{\frac{1}{4}}[\bar{w}+\ldots]+E^{\frac{1}{6}}[\tilde{w}+\ldots], \end{aligned}\right\} \tag{A 3}$$

and the boundary conditions along the side wall ($\xi = \eta = 0$) are

$$\bar{u}+\tilde{u} = \tilde{v} = \tilde{w} = 0 \quad \text{at} \quad \xi = 0. \tag{A 4}$$

It is preferable to replace the boundary condition containing $\tilde{u}$ by an equivalent, but more suitable, one. (In the steady state, the Fourier series representation of $\tilde{u}$ is known to diverge right at the side wall). This is achieved by integrating and adding the continuity equations for the $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ layers across their respective widths, viz.

$$\frac{\partial}{\partial z}\left[\int_0^\infty (E^{\frac{1}{6}}\tilde{w})(E^{\frac{1}{3}}d\xi)+\int_0^\infty (E^{\frac{1}{4}}\bar{w})(E^{\frac{1}{4}}d\eta)\right] = E^{\frac{1}{2}}(\tilde{u}+\bar{u})_{\text{sidewall}}.$$

Therefore we can replace the boundary condition on the radial velocity by

$$\int_0^\infty \tilde{w}\,d\xi+\int_0^\infty \bar{w}\,d\eta = Q, \tag{A 5}$$

where Q is as yet an arbitrary function of time only. However, by continuity arguments, it is clear that Q is proportional to the vertical mass flux in the interior; hence it is equal to zero.

Quasi-steady approximations for the $E^{\frac{1}{3}}$ fields can easily be obtained by solving he following boundary-value problem:

$$2\tilde{v}_z = \tilde{w}_{\xi\xi\xi}, \quad 2\tilde{w}_z = -\tilde{v}_{\xi\xi\xi}, \tag{A 6}$$

with

$$\left.\begin{aligned} &\tilde{v} = \tilde{w} = 0 \quad \text{for} \quad \xi = 0, \\ &\int_0^\infty \tilde{w}\,d\xi = -\int_0^\infty \bar{w}\,d\eta \quad \text{for all } z, \\ &\tilde{w} = 0 \quad \text{for} \quad z = 0, 1, \\ &\tilde{v}, \tilde{w} \to 0 \quad \text{as} \quad \xi \to \infty, \end{aligned}\right\} \tag{A 7}$$

where $\bar{w}$ is given by (18).

REFERENCES

Barcilon, V. 1967*a* *J. Mar. Res.* **25**, 1–9.

Barcilon, V. 1967*b* *J. Fluid Mech.* **27**, 551–62.

Greenspan, H. P. & Howard, L. N. 1963 *J. Fluid Mech.* **17**, 385–404.

Jacobs, S. J. 1964 *J. Fluid Mech.* **20**, 581–90.

Lewellen, W. S. 1965 *A.I.A.A.* **3**, 91–8.

Stewartson, K. 1957 *J. Fluid Mech.* **3**, 17–26.

Taylor, G. I. 1923 *Proc. Roy. Soc. Lond.* A **104**, 213–18.